Grade 6 Math Circles

March 4th-8th, 2024

The Binomial Coefficient - Problem Set

Note: Problems that are marked with * are considered challenge problems!

1. Evaluate the following factorials:
   (a) 7!
   (b) 9!
   (c) 10!

> *Solution*:
> (a) 5040
> (b) 362880
> (c) 3628800

2. Reduce the following fractions to lowest terms:
   (a) 155/225
   (b) 20/290
   (c) 252/369

> *Solution*:
> (a) 31/45
> (b) 2/29
> (c) 28/41

3. Evaluate the following quotients of factorials:
   (a) 5!/2!
   (b) 7!/9!
   (c) 12!/5!

Solution:

(a) $3 \times 4 \times 5 = 60$

(b) $1/(8 \times 9) = 1/72$

(c) $6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 = 3991680$

4. Rewrite the following products as quotients of factorials:

(a) $5 \times 4 \times 3$

(b) $8 \times 7 \times 6 \times 5$

(c) $13 \times 12 \times 11$

(d) 12

Solution:

(a) $5!/2!$

(b) $8!/4!$

(c) $13!/10!$

(d) $12!/11!$

5. Evaluate the following binomial coefficients:

(a) $\binom{5}{2}$

(b) $\binom{5}{3}$

(c) $\binom{10}{4}$

(d) $\binom{11}{7}$

Solution:

(a) 10

(b) 10

(c) 210

(d) 330

6. In how many ways can you select 5 distinct balls from a box contain 12 balls total?

Solution: The number of ways you select 5 distinct balls from a box contain 12 balls total is given by $\binom{12}{5} = 792$.

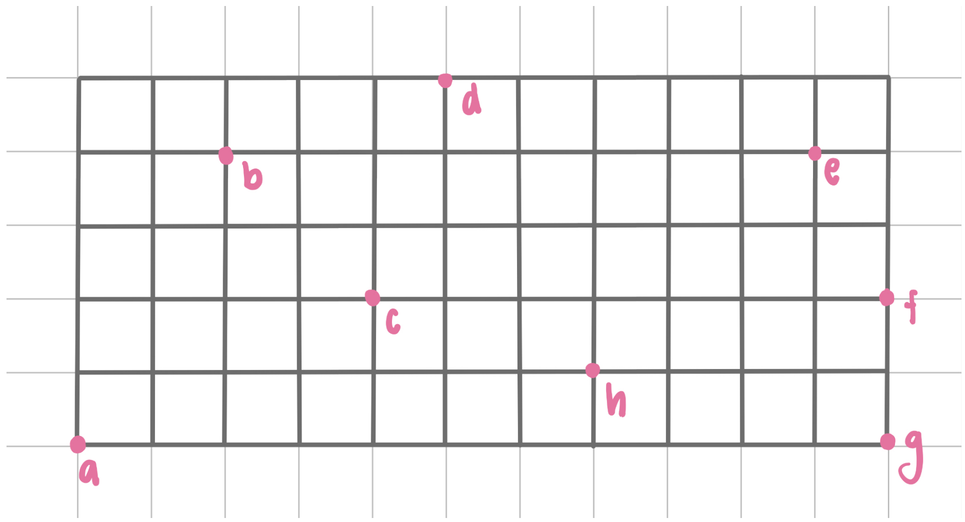
7. There is a class of 20 students and they need to select a committee of 5 students to plan a party, once these 5 students are picked one needs to be selected to be president of the committee, in how many ways can this be done?

Solution: First we need to **choose** 5 students from a class of 20 students this will be given by $\binom{20}{5} = 15504$. Now that we have chosen the five students it remains to **choose** 1 student from the chosen group of five the will be given by $\binom{5}{1} = 5$. Therefore we can select a committee of 5 people from a class of 20 and a president from the committee of 5 in $15504 \times 5 = 77520$ ways.

8. There are 12 boys and 18 girls who are eligible to run in a mixed relay. In how many ways could the relay be chosen and they run in a race if the team must contain 2 boys and 2 girls?

Solution: To build our relay team we need to **choose** 2 boys from our 12 eligible boys, this will be given by $\binom{12}{2} = 66$. In the same way we need to **choose** 2 girls from our eligible 18 girls, this will be given by $\binom{18}{2} = 153$. Therefore we can select a relay team containing 2 boys and 2 girls from 12 boys and 18 girls eligible runners in $66 \times 153 = 10098$ ways.

9. * Calculate the number of paths from the given pairs of points which which take only take steps right and up *(hint for some of these problems you will have to redraw the grid so that you only move in the right and up directions)*, in the grid below.



(a) What is the number of paths from a to d?

(b) What is the number of paths from a to f?

(c) What is the number of paths from c to e?

(d) What is the number of paths from h to e?

(e) What is the number of paths from f to b?

(f) What is the number of paths from g to h?

(g) What is the number of paths from c to g?

(h) What is the number of paths from a to g?

Solution:

(a) d is located 5 steps right and 5 steps up from a so we know we need to travel a total of 10 steps and we need to **choose** 5 of them to be right-steps, this gives us $\binom{10}{5} = 252$ distinct paths from a to d.

(b) f is located 11 steps right and 2 steps up from a so we know we need to travel a total of 13 steps and we need to **choose** 11 of them to be right-steps, this gives us $\binom{13}{11} = 78$ distinct paths from a to f.

(c) e is located 6 steps right and 2 steps up from c so we know we need to travel a total of 8 steps and we need to **choose** 6 of them to be right-steps, this gives us $\binom{8}{6} = 28$ distinct paths from c to e.

(d) e is located 3 steps right and 3 steps up from h so we know we need to travel a total of 6 steps and we need to **choose** 3 of them to be right-steps, this gives us $\binom{6}{3} = 20$ distinct paths from h to e.

(e) Redrawing our grid we can see that b is located 9 steps right and 2 steps up from f so we know we need to travel a total of 11 steps and we need to **choose** 9 of them to be right-steps, this gives us $\binom{11}{9} = 55$ distinct paths from f to b.

(f) Redrawing our grid we can see that h is located 4 steps right and 1 steps up from g so we know we need to travel a total of 5 steps and we need to **choose** 4 of them to be right-steps, this gives us $\binom{5}{4} = 5$ distinct paths from g to h.

(g) Redrawing our grid we can see that g is located 7 steps right and 2 steps up from c so we know we need to travel a total of 9 steps and we need to **choose** 7 of them to be right-steps, this gives us $\binom{9}{7} = 36$ distinct paths from c to g.

(h) g is located 11 steps right and 0 steps up from a so we know we need to travel a total of 11 steps and we need to **choose** 11 of them to be right-steps, this gives us $\binom{11}{11} = 1$ distinct path from a to g.

10. ** Using algebraic manipulations on the definition of the binomial coefficient show that $k \times \binom{n}{k} = n \times \binom{n-1}{k-1}$

Solution: We will start by writing the left hand side of the expression in lowest terms. From the definition of the binomial coefficient we know that $\binom{n}{k} = \frac{n!}{(n-k)! \times k!}$. Then $k \times \binom{n}{k}$ is equal to $\binom{n}{k} = \frac{k \times n!}{(n-k)! \times k!}$. We know $k! = 1 \times 2 \times ... \times k$, so this means we can reduce our fraction by a factor of k since we also have k in the numerator of our expression. Once we reduce by a factor of k we have $k \times \binom{n}{k} = \frac{k \times n!}{(n-k)! \times k!} = \frac{n!}{(n-k)! \times (k-1)!}$.

Let's now look at the right hand side of the expression. Once again from the definition of the binomial coefficient we know that $\binom{n-1}{k-1} = \frac{(n-1)!}{(n-1-(k-1))! \times (k-1)!} =$

$\frac{(n-1)!}{(n-k-)! \times (k-1)!}$. From this it follows that $n \times \binom{n-1}{k-1} = \frac{n \times (n-1)!}{(n-k)! \times (k-1)!}$ but $n \times (n-1)! = n!$ so we get $n \times \binom{n-1}{k-1} = \frac{n \times (n-1)!}{(n-k)! \times (k-1)!} = \frac{n!}{(n-k)! \times (k-1)!}$ which is exactly what we showed the right hand side is equal to! Therefore
$k \times \binom{n}{k} = n \times \binom{n-1}{k-1}$

11. *** Using algebraic manipulations on the definition of the binomial coefficient show that $\binom{n}{k+1} = \binom{n}{k} \times \frac{n-k}{k+1}$

Solution: We will start by writing the left hand side of the expression using the formal definition of the binomial coefficient. From the definition we know that $\binom{n}{k+1} = \frac{n!}{(n-(k+1))! \times (k+1)!} = \frac{n!}{(n-k-1))! \times (k+1)!}$.

Let's now look at the right hand side of the expression. Once again from the definition of the binomial coefficient we know that $\binom{n}{k} = \frac{n!}{(n-k)! \times k!}$. From this it follows that $\binom{n}{k} \times \frac{n-k}{k+1} = \frac{n!}{(n-k)! \times k!} \times \frac{n-k}{k+1} = \frac{n! \times (n-k)}{(n-k)! \times k! \times (k+1)}$. We know that $(n-k)! = 1 \times 2 \times ...(n-k)$, so this means we can reduce our fraction by a factor of $n-k$ since we also have $n-k$ in the numerator of our expression. Once we reduce by a factor of $n-k$ we have $\frac{n! \times (n-k)}{(n-k)! \times k! \times (k+1)} = \frac{n!}{(n-k-1)! \times k! \times (k+1)}$. The last simplification we can preform on our fraction is to notice that $k! \times k+1 = (k+1)!$, and so we get $\frac{n!}{(n-k-1)! \times k! \times (k+1)} = \frac{n!}{(n-k-1)! \times (k+1)!}$ which is exactly what we showed the right hand side is equal to! Therefore
$\binom{n}{k+1} = \binom{n}{k} \times \frac{n-k}{k+1}$.